J. Math. Anal. Appl. 345 (2008) 135–146

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Comparison of means generated by two functions and a measure ☆

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ARTICLE INFO

Article history:
Received 28 November 2007
Available online 8 April 2008
Submitted by M. Laczkovich

Keywords:
Generalized integral means
Generalized Gini means
Comparison problem

ABSTRACT

Given two continuous functions $f, g : I \to \mathbb{R}$ such that g is positive and f/g is strictly monotone, and a probability measure μ on the Borel subsets of $[0,1]$, the two variable mean $M_{f,g;\mu} : I^2 \to I$ is defined by

$$M_{f,g;\mu}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)\,d\mu(t)}{\int_0^1 g(tx+(1-t)y)\,d\mu(t)}\right) \quad (x,y \in I).$$

The aim of this paper is to study the comparison problem of these means, i.e., to find conditions for the generating functions (f,g) and (h,k) and for the measures μ, ν such that the comparison inequality

$$M_{f,g;\mu}(x,y) \leqslant M_{h,k;\nu}(x,y) \quad (x,y \in I)$$

holds.

1. Introduction

Throughout this paper I will stand for a nonempty open real interval. The classes of continuous strictly monotone and continuous positive real-valued functions defined on I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively.

In general, a continuous function $M : I^2 \to I$ is called a *two-variable mean* on I if the so-called mean value inequality

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y) \quad (x,y \in I)$$

holds. The arithmetic and geometric means are well-known instances for strict means on $\mathbb{R}_+$.

In this paper, we consider a general class of means. Given two continuous functions $f, g : I \to \mathbb{R}$ with $g \in \mathcal{CP}(I)$, $f/g \in \mathcal{CM}(I)$ and a probability measure μ on the Borel subsets of $[0,1]$, the two variable mean $M_{f,g;\mu} : I^2 \to I$ is defined by

$$M_{f,g;\mu}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)\,d\mu(t)}{\int_0^1 g(tx+(1-t)y)\,d\mu(t)}\right) \quad (x,y \in I).$$

If $\mu = \frac{\delta_0+\delta_1}{2}$ (where δ_s denotes the Dirac measure concentrated at $s \in [0,1]$), $\varphi \in \mathcal{CM}(I)$, and $p \in \mathcal{CP}(I)$, then

$$M_{p\varphi,p;\mu}(x,y) = \varphi^{-1}\left(\frac{p(x)\varphi(x)+p(y)\varphi(y)}{p(x)+p(y)}\right) \quad (x,y \in I),$$

☆ This research has been supported by the Hungarian Scientific Research Fund (OKTA) Grants K-62316, NK-68040.

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0022-247X/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2008.04.004

which was introduced and studied by Bajraktarević [1,2]. In the particular case $p = 1$, it follows that

$$M_{\varphi,1;\mu}(x, y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \quad (x, y \in I),$$

which is the well-known quasi-arithmetic mean (cf. [16]).

If μ is the Lebesgue measure on $[0, 1]$ and $\varphi, \psi : I \to \mathbb{R}$ are continuously differentiable functions with $\psi' \in \mathcal{CP}(I)$ and $\varphi'/\psi' \in \mathcal{CM}(I)$, then, by the Fundamental Theorem of Calculus, one can easily see that

$$M_{\varphi',\psi';\mu}(x, y) = \begin{cases} \left(\frac{\varphi'}{\psi'}\right)^{-1}\left(\frac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)}\right) & \text{if } x \neq y, \\ x & \text{if } x = y \end{cases} \quad (x, y \in I),$$

which is called a Cauchy or difference mean in the literature (cf. [23]). When $\psi(x) = x$, then this mean goes over into a Lagrangian mean (cf. [3,4]):

$$M_{\varphi',1;\mu}(x, y) = \begin{cases} (\varphi')^{-1}\left(\frac{\varphi(y) - \varphi(x)}{y - x}\right) & \text{if } x \neq y, \\ x & \text{if } x = y \end{cases} \quad (x, y \in I).$$

The aim of this paper is to study the general *comparison problem*

$$M_{f,g;\mu}(x, y) \leqslant M_{h,k;\nu}(x, y) \quad (x, y \in I). \tag{1}$$

We give necessary conditions (which, in general, are not sufficient) and also sufficient conditions (that are also necessary in a certain sense) for (1) to hold. Various particular cases of this problem have been studied in the papers [7–9,12,24,25, 27,32,34]. In the last section, we consider generalized Gini means that also includes Stolarsky means and their comparison problems.

2. Invariants with respect to equality of means

In order to describe the regularity conditions related to the two generating functions f, g of the mean $M_{f,g;\mu}$, we introduce some regularity classes. The class $\mathcal{C}_0(I)$ consists of all those pairs of continuous functions $f, g : I \to \mathbb{R}$ such that $g \in \mathcal{CP}(I)$ and $f/g \in \mathcal{CM}(I)$. For $(f, g) \in \mathcal{C}_0(I)$, we define the *deviation* function $\mathcal{D}_{f,g} : I^2 \to \mathbb{R}$ by

$$\mathcal{D}_{f,g}(x, y) := \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} = g(x)g(y)\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right) \quad (x, y \in I). \tag{2}$$

(In fact, the function $\mathcal{D}_{f,g}$ is a *quasi-deviation* in the sense of the paper [28].) Clearly, we have that $\mathcal{D}_{f,g}(x, y) = 0$ if and only if $x = y$. Moreover, if f/g is increasing, then $\mathcal{D}_{f,g}(x, y) \lesseqqgtr 0$ if and only if $x \lesseqqgtr y$.

If $n \geqslant 1$, then we say that the pair (f, g) is in the class $\mathcal{C}_n(I)$ if f, g are n-times continuously differentiable functions such that $g \in \mathcal{CP}(I)$ and the *Wronski determinant*

$$\begin{vmatrix} f'(x) & f(x) \\ g'(x) & g(x) \end{vmatrix} = \partial_1 \mathcal{D}_{f,g}(x, x) = g^2(x)\left(\frac{f(x)}{g(x)}\right)' \quad (x \in I) \tag{3}$$

does not vanish on I. Obviously, the latter condition implies that f/g is strictly monotone, i.e., $f/g \in \mathcal{CM}(I)$ and hence $\mathcal{C}_0(I) \supseteq \mathcal{C}_1(I) \supseteq \mathcal{C}_2(I) \supseteq \cdots$.

The next result characterizes the mean $M_{f,g;\mu}$ via an implicit equation and signifies the role of the deviation function $\mathcal{D}_{f,g}$.

Lemma 1. *Let $(f, g) \in \mathcal{C}_0(I)$ and μ be a Borel probability measure on $[0, 1]$ and assume that f/g is increasing. Then for all $x, y \in I$ and $u \in [x, y]$,*

$$M_{f,g;\mu}(x, y) \lesseqqgtr u \quad \textit{if and only if} \quad \int_0^1 \mathcal{D}_{f,g}\big(tx + (1 - t)y, u\big)\, d\mu(t) \lesseqqgtr 0. \tag{4}$$

If f/g is decreasing, then in the second inequality the inequality signs should be reversed. As a consequence of (4), we have the identity

$$\int_0^1 \mathcal{D}_{f,g}\big(tx + (1 - t)y, M_{f,g;\mu}(x, y)\big)\, d\mu(t) = 0 \quad (x, y \in I). \tag{5}$$

Proof. Expanding the determinant $\mathcal{D}_{f,g}(tx+(1-t)y,u)$ by the second column, the second inequality of (4) can equivalently be written as

$$g(u)\int_0^1 f\big(tx+(1-t)y\big)\,d\mu(t) \lesseqqgtr f(u)\int_0^1 g\big(tx+(1-t)y\big)\,d\mu(t).$$

Dividing by $g(u)\int_0^1 g(tx+(1-t)y)\,d\mu(t)>0$, and applying the inverse function of f/g to both sides, it follows that $M_{f,g;\mu}(x,y) \lesseqqgtr u$. $\square$

It is easy to check (e.g., by using Lemma 1) that the identity

$$M_{f,g;\mu} = M_{f^*,g^*;\mu} \tag{6}$$

holds if

$$f=\alpha f^*+\beta g^*, \qquad g=\gamma f^*+\delta g^*, \tag{7}$$

where the constants $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ satisfy $\alpha\delta-\beta\gamma\neq 0$. According to the results in [22] and [26], for certain measures μ, (7) is the principal solution of the equality problem (6). If (7) holds, then we say that the pairs (f,g) and (f^*,g^*) are equivalent.

It is obvious that any necessary and/or sufficient condition for (1) has to be invariant with respect to the equivalence of the generating functions. On the other hand, it is easy to see that if $(f,g),(f^*,g^*)\in\mathcal{C}_0(I)$ and (7) holds for some constants α, β, γ, δ, then

$$\mathcal{D}_{f,g}(x,y)=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\cdot\mathcal{D}_{f^*,g^*}(x,y) \quad (x,y\in I).$$

Therefore, the ratios of the deviation functions and their partial derivatives are invariant with respect to the equivalence of generating functions. In the sequel, we shall consider (under the indicated regularity assumptions on the generating functions f,g) the following three basic invariants:

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{f,g}(x,u)}=\frac{\begin{vmatrix} f(u) & f(y)\\ g(u) & g(y)\end{vmatrix}}{\begin{vmatrix} f(x) & f(u)\\ g(x) & g(u)\end{vmatrix}} \quad (u,x,y\in I,\ x<u<y),\ (f,g)\in\mathcal{C}_0(I),$$

$$\frac{\mathcal{D}_{f,g}(x,y)}{\partial_1\mathcal{D}_{f,g}(y,y)}=\frac{\begin{vmatrix} f(x) & f(y)\\ g(x) & g(y)\end{vmatrix}}{\begin{vmatrix} f'(y) & f(y)\\ g'(y) & g(y)\end{vmatrix}} \quad (x,y\in I),\ (f,g)\in\mathcal{C}_1(I),$$

$$\frac{\partial_1^2\mathcal{D}_{f,g}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)}=\frac{\begin{vmatrix} f''(x) & f(x)\\ g''(x) & g(x)\end{vmatrix}}{\begin{vmatrix} f'(x) & f(x)\\ g'(x) & g(x)\end{vmatrix}} \quad (x\in I),\ (f,g)\in\mathcal{C}_2(I).$$

The third invariant first appeared in a 1969 paper of Bajraktarević [2] in the form

$$\frac{\varphi''(x)}{\varphi'(x)}+2\frac{g'(x)}{g(x)} \quad (x\in I),$$

where $\varphi(x)=f(x)/g(x)$ $(x\in I)$. The second invariant was found by Daróczy and Losonczi in 1971 [12], written in the form

$$\frac{\varphi(x)-\varphi(y)}{\varphi'(y)}\,\frac{g(x)}{g(y)} \quad (x,y\in I),$$

while the first invariant first appeared in a 1982 paper of Daróczy and Páles [13] in the form

$$\frac{E(u,y)}{E(x,u)} \quad (u,x,y\in I,\ x<u<y),$$

where $E:I\times I\to I$ is a deviation function.

Of course all these three invariants turned up in a number of papers dealing with functional equations and inequalities for various classes of mean values (see e.g. [10,11,14,18–21,29,30]). The first invariant is connected to the one-parameter family of means $M_{f,g;m_s}$, where $(m_s)_{s\in[0,1]}$ is the one-parameter family of measures defined by

$$m_s:=(1-s)\delta_0+s\delta_1 \quad \big(s\in[0,1]\big). \tag{8}$$

Lemma 2. *Let $(f,g) \in \mathcal{C}_0(I)$. Then, for all fixed $x, y \in I$ with $x < y$, the values $u \in]x, y[$ and $s \in]0,1[$ satisfy*

$$M_{f,g;m_s}(x,y) \lesseqqgtr u \quad \text{if and only if} \quad \frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{f,g}(x,u)} \lesseqqgtr \frac{s}{1-s}. \tag{9}$$

Proof. Observe that, for $x, y \in I$ and $s \in [0,1]$,

$$M_{f,g;m_s}(x,y) = \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)\,d[(1-s)\delta_0+s\delta_1](t)}{\int_0^1 g(tx+(1-t)y)\,d[(1-s)\delta_0+s\delta_1](t)}\right) = \left(\frac{f}{g}\right)^{-1}\left(\frac{(1-s)f(y)+sf(x)}{(1-s)g(y)+sg(x)}\right). \tag{10}$$

To prove (9), assume that f/g is increasing and let $x, y \in I$ be fixed with $x < y$. Using (10), we can see that the inequality $M_{f,g;m_s}(x,y) \leqslant u$ is equivalent to

$$\frac{sf(x)+(1-s)f(y)}{sg(x)+(1-s)g(y)} \lesseqqgtr \frac{f(u)}{g(u)},$$

which can be rewritten as

$$s\big[f(x)g(u)-f(u)g(x)\big] \lesseqqgtr (1-s)\big[f(u)g(y)-f(y)g(u)\big].$$

By (2) and the increasingness of f/g, we have $\mathcal{D}_{f,g}(u,y) < 0$, hence the latter inequality is equivalent to

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{f,g}(x,u)} \lesseqqgtr \frac{s}{1-s}.$$

In the case when f/g is decreasing, the proof is analogous. $\square$

The second invariant is motivated by the next result.

Lemma 3. *Let $(f,g) \in \mathcal{C}_1(I)$. Then*

$$\lim_{s\to 0} \frac{1}{s}\big[M_{f,g;m_s}(x,y)-y\big] = \frac{\mathcal{D}_{f,g}(x,y)}{\partial_1\mathcal{D}_{f,g}(y,y)} \quad (x, y \in I). \tag{11}$$

Proof. Using formula (10) and L'Hospital's Rule, the limit on the leftt-hand side of (11) can be computed in the following way:

$$\begin{aligned}
\lim_{s\to 0} \frac{1}{s}\big[M_{f,g;m_s}(x,y)-y\big] &= \lim_{s\to 0} \frac{1}{s}\left[\left(\frac{f}{g}\right)^{-1}\left(\frac{sf(x)+(1-s)f(y)}{sg(x)+(1-s)g(y)}\right)-y\right] \\
&= \lim_{s\to 0}\left(\left(\frac{f}{g}\right)^{-1}\right)'\left(\frac{sf(x)+(1-s)f(y)}{sg(x)+(1-s)g(y)}\right)\cdot\frac{f(x)g(y)-g(x)f(y)}{[sg(x)+(1-s)g(y)]^2} \\
&= \left(\left(\frac{f}{g}\right)^{-1}\right)'\left(\frac{f(y)}{g(y)}\right)\cdot\frac{f(x)g(y)-g(x)f(y)}{g^2(y)} \\
&= \frac{g^2(y)}{f'(y)g(y)-f(y)g'(y)}\cdot\frac{f(x)g(y)-g(x)f(y)}{g^2(y)} = \frac{\mathcal{D}_{f,g}(x,y)}{\partial_1\mathcal{D}_{f,g}(y,y)}. \qquad \square
\end{aligned}$$

The third invariant is connected to the second-order partial derivatives of the mean $M_{f,g;\mu}$, as we have the following

Lemma 4. *Let $(f,g) \in \mathcal{C}_2(I)$ and μ be a Borel probability measure on $[0,1]$. Then $M_{f,g;\mu}$ is two times continuously differentiable on I^2 and*

$$\begin{aligned}
&\partial_1 M_{f,g;\mu}(x,x) = \mu_1 \quad (x \in I), \\
&\partial_1^2 M_{f,g;\mu}(x,x) = \big(\mu_2-\mu_1^2\big)\frac{\partial_1^2\mathcal{D}_{f,g}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)} \quad (x \in I),
\end{aligned} \tag{12}$$

where

$$\mu_1 := \int_0^1 t\,d\mu(t) \quad \text{and} \quad \mu_2 := \int_0^1 t^2\,d\mu(t) \tag{13}$$

are the first and second moments of the measure μ.

Proof. The twice continuous differentiability of $M_{f,g;\mu}$ is the consequence of the standard calculus rules and the differentiability of the inverse function of f/g. Differentiating the identity (5) with respect to x once and twice, and then substituting $y := x$, we get

$$\int_0^1 \left[\partial_1\mathcal{D}_{f,g}(x,x)t + \partial_2\mathcal{D}_{f,g}(x,x)\partial_1 M_{f,g;\mu}(x,x)\right]d\mu(t) = 0$$

and

$$\int_0^1 \left[\partial_1^2\mathcal{D}_{f,g}(x,x)t^2 + 2\partial_1\partial_2\mathcal{D}_{f,g}(x,x)\partial_1 M_{f,g;\mu}(x,x)t + \partial_2^2\mathcal{D}_{f,g}(x,x)\big(\partial_1 M_{f,g;\mu}(x,x)\big)^2 + \partial_2\mathcal{D}_{f,g}(x,x)\partial_1^2 M_{f,g;\mu}(x,x)\right]d\mu(t) = 0,$$

respectively. Using the identities $\partial_2\mathcal{D}_{f,g}(x,x) = -\partial_1\mathcal{D}_{f,g}(x,x)$, $\partial_2^2\mathcal{D}_{f,g}(x,x) = -\partial_1^2\mathcal{D}_{f,g}(x,x)$, and $\partial_1\partial_2\mathcal{D}_{f,g}(x,x) = 0$ (that are consequences of the asymmetry property $\mathcal{D}_{f,g}(x,y) = -\mathcal{D}_{f,g}(y,x)$), we get that

$$\partial_1\mathcal{D}_{f,g}(x,x)\left[\mu_1 - \partial_1 M_{f,g;\mu}(x,x)\right] = 0$$

and

$$\partial_1^2\mathcal{D}_{f,g}(x,x)\left[\mu_2 - \big(\partial_1 M_{f,g;\mu}(x,x)\big)^2\right] - \partial_1\mathcal{D}_{f,g}(x,x)\partial_1^2 M_{f,g;\mu}(x,x) = 0.$$

By $\partial_1\mathcal{D}_{f,g}(x,x) \neq 0$, these equalities yield the formulae for the first- and second-order partial derivatives in (12). □

In terms of the three invariants introduced above necessary conditions and also sufficient conditions will be given for the comparison and subadditivity problem of the general means $M_{f,g;\mu}$ in the subsequent sections.

3. Necessary conditions, sufficient conditions

Our first result is a necessary condition for the comparison inequality (1). The strict form of this condition is also sufficient for the local comparability.

Theorem 5. *Let $(f,g), (h,k) \in \mathcal{C}_2(I)$ and μ, ν be Borel probability measures on $[0,1]$. Suppose that*

$$M_{f,g;\mu}(x,y) \leqslant M_{h,k;\nu}(x,y) \quad (x, y \in I) \tag{1}$$

holds. Then

$$\mu_1 = \nu_1 \tag{14}$$

and

$$\big(\mu_2 - \mu_1^2\big)\frac{\partial_1^2\mathcal{D}_{f,g}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)} \leqslant \big(\nu_2 - \nu_1^2\big)\frac{\partial_1^2\mathcal{D}_{h,k}(x,x)}{\partial_1\mathcal{D}_{h,k}(x,x)} \quad (x \in I). \tag{15}$$

The latter inequality is equivalent to the increasingness of the function

$$x \mapsto \frac{|\partial_1\mathcal{D}_{h,k}(x,x)|^{\nu_2-\nu_1^2}}{|\partial_1\mathcal{D}_{f,g}(x,x)|^{\mu_2-\mu_1^2}} \quad (x \in I). \tag{16}$$

Conversely, if (14) *is valid and* (15) *is satisfied with strict inequality sign, then* (1) *holds for $x, y \in I$ provided that $|x-y|$ is small enough.*

Proof. To see that inequality (15) is equivalent to the increasingness of the function (16), compute the derivative of (16) using the identities

$$\frac{d}{dx}\partial_1\mathcal{D}_{f,g}(x,x) = \partial_1^2\mathcal{D}_{f,g}(x,x), \qquad \frac{d}{dx}\partial_1\mathcal{D}_{h,k}(x,x) = \partial_1^2\mathcal{D}_{h,k}(x,x) \quad (x \in I),$$

and check that its nonnegativity can be written as (15).

For a fixed $y \in I$, the function

$$D(x) = M_{h,k;\nu}(x,y) - M_{f,g;\mu}(x,y) \quad (x \in I) \tag{17}$$

is nonnegative by (1) and attains its minimum at $x = y$. Therefore $D'(y) = 0$ and $D''(y) \geqslant 0$. Applying Lemma 4, the necessity of conditions (14) and (15) follows.

Now assume that (14) holds and (15) is satisfied with strict inequality sign. By Taylor's formulae with second-order remainder term, for a fixed $y \in I$, with the notation (17), we have

$$D(x) = D(y) + D'(y)(x-y) + \frac{1}{2}D''(\xi)(x-y)^2 = \frac{1}{2}D''(\xi)(x-y)^2 \quad (x \in I)$$

where ξ is between y and x. Since $D''(y) > 0$, therefore, by a continuity argument, $D''(\xi) \geqslant 0$ if x and y are near enough. □

In the sequel, we consider the case when $\mu = \nu$. In what follows, we give a condition containing two independent variables for (1) which does not involve the measure μ and assumes first-order continuous differentiability.

Theorem 6. *Let $(f, g), (h, k) \in \mathcal{C}_1(I)$. The following three assertions are equivalent*:

(i) *for all Borel probability measures μ on* $[0, 1]$,

$$M_{f,g;\mu}(x, y) \leqslant M_{h,k;\mu}(x, y) \quad (x, y \in I); \tag{18}$$

(ii) *for all $i \in \mathbb{N}$*

$$M_{f,g;m_{\frac{1}{i}}}(x, y) \leqslant M_{h,k;m_{\frac{1}{i}}}(x, y) \quad (x, y \in I), \tag{19}$$

(*where (m_s) is the one-parameter family of measures defined by* (8));

(iii)

$$\frac{\mathcal{D}_{f,g}(v, u)}{\partial_1 \mathcal{D}_{f,g}(u, u)} \leqslant \frac{\mathcal{D}_{h,k}(v, u)}{\partial_1 \mathcal{D}_{h,k}(u, u)} \quad (u, v \in I). \tag{20}$$

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. To prove (ii) $\Rightarrow$ (iii), use (19) and Lemma 3 to get

$$\frac{\mathcal{D}_{f,g}(x, y)}{\partial_1 \mathcal{D}_{f,g}(y, y)} = \lim_{i\to\infty} i\big[M_{f,g;m_{\frac{1}{i}}}(x, y) - y\big] \leqslant \lim_{i\to\infty} i\big[M_{h,k;m_{\frac{1}{i}}}(x, y) - y\big] = \frac{\mathcal{D}_{h,k}(x, y)}{\partial_1 \mathcal{D}_{h,k}(y, y)} \quad (x, y \in I),$$

which proves (20).

(iii) $\Rightarrow$ (i) Substituting

$$v := tx + (1-t)y, \qquad u := M_{h,k;\mu}(x, y) \quad (x, y \in I)$$

into (20) and integrating on $[0, 1]$, we get

$$\frac{\int_0^1 \mathcal{D}_{f,g}(tx + (1-t)y, M_{h,k;\mu}(x, y))\, d\mu(t)}{\partial_1 \mathcal{D}_{f,g}(M_{h,k;\mu}(x, y))} \leqslant \frac{\int_0^1 \mathcal{D}_{h,k}(tx + (1-t)y, M_{h,k;\mu}(x, y))\, d\mu(t)}{\partial_1 \mathcal{D}_{h,k}(M_{h,k;\mu}(x, y))}. \tag{21}$$

By Lemma 1, the numerator of the right-hand side of this inequality is zero. Assuming that f/g is strictly increasing, we have that $\partial_1 \mathcal{D}_{f,g} = g^2(f/g)' > 0$. Thus, we obtain from (21) that

$$\int_0^1 \mathcal{D}_{f,g}\big(tx + (1-t)y, M_{h,k;\mu}(x, y)\big)\, d\mu(t) \leqslant 0, \tag{22}$$

which, by Lemma 1 again, is equivalent to (18).

In the case when f/g is strictly decreasing, the proof is similar. □

The following condition for (1) contains three variables, it also does not involve the measure μ, and assumes only continuity.

Theorem 7. *Let $(f, g), (h, k) \in \mathcal{C}_0(I)$. The following three assertions are equivalent:*

(i) *for all Borel probability measures μ on* $[0, 1]$,

$$M_{f,g;\mu}(x, y) \leqslant M_{h,k;\mu}(x, y) \quad (x, y \in I); \tag{18}$$

(ii) *for all $s \in [0, 1]$*,

$$M_{f,g;m_s}(x, y) \leqslant M_{h,k;m_s}(x, y) \quad (x, y \in I) \tag{23}$$

(*where (m_s) is the one-parameter family of measures defined by* (8));

(iii)

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{f,g}(x,u)} \leqslant \frac{\mathcal{D}_{h,k}(u,y)}{\mathcal{D}_{h,k}(x,u)} \quad (x,u,y \in I,\ x<u<y). \tag{24}$$

Proof. The implication (i) ⇒ (ii) is obvious. To prove (ii) ⇒ (iii), let $x<y$ be fixed elements of I. Assume that (23) holds for all $s \in [0,1]$. Let $u \in]x,y[$ be arbitrary and define $s \in]0,1[$ by the equation

$$\frac{\mathcal{D}_{h,k}(u,y)}{\mathcal{D}_{h,k}(x,u)} = \frac{s}{1-s}.$$

Then, by the inequality (23) and by Lemma 2, we get

$$M_{f,g;m_s}(x,y) \leqslant M_{h,k;m_s}(x,y) = u.$$

Thus, again by Lemma 2, it follows that

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{f,g}(x,u)} \leqslant \frac{s}{1-s} = \frac{\mathcal{D}_{h,k}(u,y)}{\mathcal{D}_{h,k}(x,u)},$$

proving (24).

(i) ⇒ (ii) Without loss of generality, we may assume that f/g and h/k are increasing functions. Then, for a fixed $u \in I$, the determinants $\mathcal{D}_{f,g}(x,u)$ and $\mathcal{D}_{h,k}(u,y)$ are negative for all elements $x,y \in I$ with $x<u<y$. Rearranging (24), we get

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{h,k}(u,y)} \leqslant \frac{\mathcal{D}_{f,g}(x,u)}{\mathcal{D}_{h,k}(x,u)}.$$

Hence,

$$\Phi(u) := \sup_{y\in I,\ u<y} \frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{h,k}(u,y)} \leqslant \inf_{x\in I,\ x<u} \frac{\mathcal{D}_{f,g}(x,u)}{\mathcal{D}_{h,k}(x,u)},$$

which yields

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{h,k}(u,y)} \leqslant \Phi(u) \leqslant \frac{\mathcal{D}_{f,g}(x,u)}{\mathcal{D}_{h,k}(x,u)} \quad (x<u<y). \tag{25}$$

Distinguishing the cases $v<u$ and $v>u$, the second and first inequality in (25) implies that, for all $u,v \in I$,

$$\mathcal{D}_{f,g}(v,u) \leqslant \Phi(u)\mathcal{D}_{h,k}(v,u). \tag{26}$$

To prove (18), let $x,y \in I$ be fixed and substitute

$$v := tx+(1-t)y, \qquad u := M_{h,k;\mu}(x,y) \quad (x,y \in I)$$

into (26). Integrating the inequality so obtained with respect to the measure μ, we get

$$\int_0^1 \mathcal{D}_{f,g}\big(tx+(1-t)y, M_{h,k;\mu}(x,y)\big)\,d\mu(t) \leqslant \Phi\big(M_{h,k;\mu}(x,y)\big) \int_0^1 \mathcal{D}_{h,k}\big(tx+(1-t)y, M_{h,k;\mu}(x,y)\big)\,d\mu(t). \tag{27}$$

By Lemma 1, the right-hand side of this inequality is zero. Thus, by (27),

$$\int_0^1 \mathcal{D}_{f,g}\big(tx+(1-t)y, M_{h,k;\mu}(x,y)\big)\,d\mu(t) \leqslant 0,$$

which, by Lemma 1 again, is equivalent to (18). □

The following result clarifies the situation when conditions (iii) of Theorem 7, (iii) of Theorem 6, and the increasingness of the function (28) are mutually equivalent to each other.

Theorem 8. *Let $(f,g),(h,k) \in \mathcal{C}_1(I)$. The following three assertions are equivalent:*

(i)

$$\frac{\mathcal{D}_{f,g}(u,y)}{\mathcal{D}_{f,g}(x,u)} \leqslant \frac{\mathcal{D}_{h,k}(u,y)}{\mathcal{D}_{h,k}(x,u)} \quad (x,u,y \in I,\ x<u<y); \tag{24}$$

(ii)

$$\frac{\mathcal{D}_{f,g}(v,u)}{\partial_1\mathcal{D}_{f,g}(u,u)} \leqslant \frac{\mathcal{D}_{h,k}(v,u)}{\partial_1\mathcal{D}_{h,k}(u,u)} \quad (u,v\in I); \tag{20}$$

(iii)

$$x \mapsto \frac{|\partial_1\mathcal{D}_{h,k}(x,x)|}{|\partial_1\mathcal{D}_{f,g}(x,x)|} \quad (x\in I) \tag{28}$$

is increasing and the following mean value principle holds: for all $x<y$ in I, there exists $u\in\,]x,y[$ such that

$$\frac{\mathcal{D}_{h,k}(x,y)}{\mathcal{D}_{f,g}(x,y)} = \frac{\partial_1\mathcal{D}_{h,k}(u,u)}{\partial_1\mathcal{D}_{f,g}(u,u)}. \tag{29}$$

Proof. (i) $\Leftrightarrow$ (ii) Assume first that (24) holds. Multiplying (24) by $x-u$ and taking the limit $x\to u$, we obtain (20) with $v=y>u$. For the case $v<u$, divide (24) by $u-y$ and take the limit $y\to u$. Then (20) follows with $v=x<u$. Conversely, suppose (20) and let $x<u<y$. Applying (20) with $v=x$ and $v=y$, we get

$$\frac{\mathcal{D}_{f,g}(x,u)}{\partial_1\mathcal{D}_{f,g}(u,u)} \leqslant \frac{\mathcal{D}_{h,k}(x,u)}{\partial_1\mathcal{D}_{h,k}(u,u)} \quad\text{and}\quad -\frac{\mathcal{D}_{f,g}(u,y)}{\partial_1\mathcal{D}_{f,g}(u,u)} \leqslant -\frac{\mathcal{D}_{h,k}(u,y)}{\partial_1\mathcal{D}_{h,k}(u,u)}. \tag{30}$$

By $x<u$ and $u<y$, both sides of the first and second inequality are negative and positive, respectively. Thus the first inequality can be rewritten as

$$-\frac{\partial_1\mathcal{D}_{f,g}(u,u)}{\mathcal{D}_{f,g}(x,u)} \leqslant -\frac{\partial_1\mathcal{D}_{h,k}(u,u)}{\mathcal{D}_{h,k}(x,u)}.$$

This multiplying this inequality with the second one from (30), the inequality (24) follows.

(ii) $\Leftrightarrow$ (iii) Without loss of generality, we may assume that f/g and h/k are increasing functions. Suppose that (20) holds and let $x<y$ be fixed in I. Then $\mathcal{D}_{f,g}(y,x)$ and $\partial_1\mathcal{D}_{h,k}(x,x)$ are positive. Hence (20) with $u=x$ and $v=y$ implies

$$\frac{\partial_1\mathcal{D}_{h,k}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)} \leqslant \frac{\mathcal{D}_{h,k}(y,x)}{\mathcal{D}_{f,g}(y,x)}.$$

With the substitution of $u=y$ and $v=x$ in (20), we similarly get

$$\frac{\mathcal{D}_{h,k}(x,y)}{\mathcal{D}_{f,g}(x,y)} = \frac{\mathcal{D}_{h,k}(y,x)}{\mathcal{D}_{f,g}(y,x)} \leqslant \frac{\partial_1\mathcal{D}_{h,k}(y,y)}{\partial_1\mathcal{D}_{f,g}(y,y)}.$$

It follows from these two inequalities that

$$\frac{\partial_1\mathcal{D}_{h,k}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)} \leqslant \frac{\mathcal{D}_{h,k}(x,y)}{\mathcal{D}_{f,g}(x,y)} \leqslant \frac{\partial_1\mathcal{D}_{h,k}(y,y)}{\partial_1\mathcal{D}_{f,g}(y,y)}.$$

This proves that the function (28) is increasing and, by the continuity of this function and the Bolzano mean value theorem, (29) holds for some $u\in\,]x,y[$.

Now assume (iii) and let $u,v\in I$. If $u<v$, then there exists an element $w\in\,]u,v[$ such that

$$\frac{\mathcal{D}_{h,k}(v,u)}{\mathcal{D}_{f,g}(v,u)} = \frac{\partial_1\mathcal{D}_{h,k}(w,w)}{\partial_1\mathcal{D}_{f,g}(w,w)}.$$

By the increasingness of the function (28), we have

$$\frac{\partial_1\mathcal{D}_{h,k}(u,u)}{\partial_1\mathcal{D}_{f,g}(u,u)} \leqslant \frac{\partial_1\mathcal{D}_{h,k}(w,w)}{\partial_1\mathcal{D}_{f,g}(w,w)} = \frac{\mathcal{D}_{h,k}(v,u)}{\mathcal{D}_{f,g}(v,u)}$$

which, after rearrangements, yields (20). The proof of (20) for the case $v<u$ is analogous. □

In certain particular cases, for a fixed measure μ, the necessary condition provided by Theorem 5 turns out to be also sufficient.

Theorem 9. *Let $(f,g),(h,k)\in\mathcal{C}_2(I)$ with $g=k$ and μ be a Borel probability measure such that $\mu_2-\mu_1^2\neq 0$. Then the comparison inequality*

$$M_{f,g;\mu}(x,y) \leqslant M_{h,k;\mu}(x,y) \quad (x,y\in I) \tag{18}$$

holds if and only if $\left(\frac{h}{k}\right)\circ\left(\frac{f}{g}\right)^{-1}$ is convex (concave) provided that the function $\frac{h}{k}$ is increasing (decreasing).

Proof. Without loss of generality, we may assume that $\frac{f}{g}$ and $\frac{h}{k}$ are increasing functions.

Suppose first that (18) holds. By Theorem 5, we have that the function

$$x \mapsto \left|\frac{\partial_1 \mathcal{D}_{h,k}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)}\right|^{\mu_2-\mu_1^2} \quad (x \in I) \tag{31}$$

is increasing. However, by $\mu_2 - \mu_1^2 \neq 0$ and

$$\mu_2 - \mu_1^2 = \int_0^1 (t-\mu_1)^2\, d\mu(t) \geqslant 0$$

it follows that $\mu_2 - \mu_1^2 > 0$. Thus, the function

$$x \mapsto \frac{\partial_1 \mathcal{D}_{h,k}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)} = \frac{\partial_1 \mathcal{D}_{h,g}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)} = \frac{\left(\frac{h}{g}\right)'(x)}{\left(\frac{f}{g}\right)'(x)} = \frac{\left(\frac{h}{k}\right)'(x)}{\left(\frac{f}{g}\right)'(x)} = \left(\left(\frac{h}{k}\right) \circ \left(\frac{f}{g}\right)^{-1}\right)' \circ \left(\frac{f}{g}\right)(x)$$

is increasing, too. Therefore, $\left(\left(\frac{h}{k}\right) \circ \left(\frac{f}{g}\right)^{-1}\right)'$ is increasing which is equivalent to the convexity of $\left(\frac{h}{k}\right) \circ \left(\frac{f}{g}\right)^{-1}$.

Conversely, suppose that $\left(\frac{h}{k}\right) \circ \left(\frac{f}{g}\right)^{-1}$ is convex. Then, as we have seen above, the function $x \mapsto \frac{\partial_1 \mathcal{D}_{h,k}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)}$ is increasing. We show that the mean value principle of Theorem 8 holds.

Indeed, using the Cauchy mean value theorem, for all $x < y$ in I, there exists a point $u \in]x, y[$, such that

$$\frac{\mathcal{D}_{h,k}(x,y)}{\mathcal{D}_{f,g}(x,y)} = \frac{\mathcal{D}_{h,g}(x,y)}{\mathcal{D}_{f,g}(x,y)} = \frac{\left(\frac{h}{g}\right)(x) - \left(\frac{h}{g}\right)(y)}{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(y)} = \frac{\left(\frac{h}{g}\right)'(u)}{\left(\frac{f}{g}\right)'(u)} = \frac{\partial_1 \mathcal{D}_{h,g}(u,u)}{\partial_1 \mathcal{D}_{f,g}(u,u)} = \frac{\partial_1 \mathcal{D}_{h,k}(u,u)}{\partial_1 \mathcal{D}_{f,g}(u,u)},$$

verifying the mean value principle.

Thus, we have proved that condition (iii) of Theorem 8 is satisfied and hence conditions (i) and (ii) (that are identical to conditions (iii) of Theorem 6 and Theorem 7, respectively) are also valid. Thus, by these theorems, the comparison inequality (18) holds. □

In the particular case when $g = h = 1$, we immediately get a comparison theorem for generalized quasi-arithmetic means.

Corollary 10. *Let $f, h : I \to \mathbb{R}$ be twice continuously differentiable functions with non-vanishing first derivatives and let μ be a Borel probability measure such that $\mu_2 - \mu_1^2 \neq 0$. Then the comparison inequality*

$$f^{-1}\left(\int_0^1 f\big(tx + (1-t)y\big)\, d\mu(t)\right) \leqslant h^{-1}\left(\int_0^1 h\big(tx + (1-t)y\big)\, d\mu(t)\right) \quad (x, y \in I)$$

holds if and only if $h \circ f^{-1}$ is convex (concave) provided that the function h is increasing (decreasing).

Theorem 11. *Let $(f, g), (h, k) \in \mathcal{C}_2(I)$ with $\frac{f}{g} = \frac{h}{k}$ and μ be a Borel probability measure such that $\mu_2 - \mu_1^2 \neq 0$. Then the comparison inequality*

$$M_{f,g;\mu}(x,y) \leqslant M_{h,k;\mu}(x,y) \quad (x, y \in I); \tag{18}$$

holds if and only if the function $\frac{k}{g}$ is increasing.

Proof. Without loss of generality, we again may assume that $\varphi := \frac{f}{g} = \frac{h}{k}$ is an increasing function.

Suppose first that (18) holds. Arguing as in the previous proof, we have that the function

$$x \mapsto \frac{\partial_1 \mathcal{D}_{h,k}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)} = \frac{\partial_1 \mathcal{D}_{\varphi k,k}(x,x)}{\partial_1 \mathcal{D}_{\varphi g,g}(x,x)} = \left(\frac{k}{g}\right)^2(x) \tag{32}$$

is increasing, which proves the increasingness of $\frac{k}{g}$.

Conversely, suppose that $\frac{k}{g}$ is increasing. Then, as we can see from (32), the function $x \mapsto \frac{\partial_1 \mathcal{D}_{h,k}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)}$ is increasing. We show that the mean value principle of Theorem 8 holds.

Indeed, using the Bolzano mean value theorem (and the continuity of the function $\frac{k}{g}$), for all $x < y$, there exists a point $u \in]x, y[$, such that

$$\frac{\mathcal{D}_{h,k}(x,y)}{\mathcal{D}_{f,g}(x,y)} = \frac{\mathcal{D}_{\varphi k,k}(x,y)}{\mathcal{D}_{\varphi g,g}(x,y)} = \frac{k(x)}{g(x)}\frac{k(y)}{g(y)} = \left(\frac{k}{g}\right)^2(u) = \frac{\partial_1 \mathcal{D}_{\varphi k,k}(u,u)}{\partial_1 \mathcal{D}_{\varphi g,g}(u,u)} = \frac{\partial_1 \mathcal{D}_{h,k}(u,u)}{\partial_1 \mathcal{D}_{f,g}(u,u)},$$

justifying the mean value principle.

Thus, we have shown that condition (iii) of Theorem 8 holds and hence conditions (i) and (ii) are also valid. Thus, in view of Theorems 6 and 7, the comparison inequality (18) is satisfied. □

4. Comparison of generalized Gini means

Consider now the setting when $I = \mathbb{R}_+$ and the functions f, g are power functions, more precisely, for $p, q \in \mathbb{R}$, define

$$\begin{aligned} &f(x) = x^p, \qquad g(x) = x^q \quad \text{if } p \neq q,\\ &f(x) = x^p \ln x, \qquad g(x) = x^p \quad \text{if } p = q. \end{aligned} \tag{33}$$

Then the mean $M_{f,g;\mu}$ reduces to the following generalization of the so-called Gini means:

$$G_{p,q;\mu}(x,y) := \begin{cases} \left(\frac{\int_0^1 (tx+(1-t)y)^p\,d\mu(t)}{\int_0^1 (tx+(1-t)y)^q\,d\mu(t)}\right)^{\frac{1}{p-q}} & \text{if } p \neq q,\\ \exp\left(\frac{\int_0^1 (tx+(1-t)y)^p \ln(tx+(1-t)y)\,d\mu(t)}{\int_0^1 (tx+(1-t)y)^p\,d\mu(t)}\right) & \text{if } p = q \end{cases} \quad (x, y \in \mathbb{R}_+).$$

In the particular case when $\mu = \frac{1}{2}(\delta_0 + \delta_1)$, the mean $G_{p,q;\mu}$ goes over into the standard Gini mean (cf. [15]) defined as

$$G_{p,q;\mu}(x,y) = G_{p,q}(x,y) := \begin{cases} \left(\frac{x^p+y^p}{x^q+y^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q,\\ \exp\left(\frac{x^p \ln x + y^p \ln y}{x^p+y^p}\right) & \text{if } p = q \end{cases} \quad (x, y \in \mathbb{R}_+).$$

The other particular case of great importance is when μ is equal to the Lebesgue measure λ. Then

$$G_{p,q;\lambda}(x,y) = S_{p+1,q+1}(x,y) \quad (x, y \in \mathbb{R}_+),$$

where $S_{p,q}$ is the so-called Stolarsky mean (cf. [35]) given by

$$S_{p,q}(x,y) := \begin{cases} \left(\frac{q(x^p-y^p)}{p(x^q-y^q)}\right)^{\frac{1}{p-q}} & \text{if } (p-q)pq \neq 0,\\ \exp\left(-\frac{1}{p} + \frac{x^p \ln x - y^p \ln y}{x^p - y^p}\right) & \text{if } p = q \neq 0,\\ \left(\frac{x^p-y^p}{p(\ln x - \ln y)}\right)^{\frac{1}{p}} & \text{if } p \neq 0,\ q = 0,\\ \left(\frac{x^q-y^q}{q(\ln x - \ln y)}\right)^{\frac{1}{q}} & \text{if } p = 0,\ q \neq 0,\\ \sqrt{xy} & \text{if } p = q = 0 \end{cases} \quad (x, y \in \mathbb{R}_+).$$

To recall the comparison theorems of Gini and Stolarsky means, introduce the following notations. For $u, v \in \mathbb{R}$, let

$$\alpha(u,v) := \begin{cases} \frac{|u|-|v|}{u-v} & \text{if } u \neq v,\\ \operatorname{sign}(u) & \text{if } u = v, \end{cases}$$

$$\beta(u,v) := \begin{cases} \min\{u,v\} & \text{if } u, v \geqslant 0,\\ 0 & \text{if } uv < 0,\\ \max\{u,v\} & \text{if } u, v \leqslant 0, \end{cases}$$

$$\gamma(u,v) := \begin{cases} \frac{u-v}{\log(u/v)} & \text{if } 0 < uv \text{ and } u \neq v,\\ u & \text{if } 0 < uv \text{ and } u = v,\\ 0 & \text{otherwise.} \end{cases}$$

The following result solves the comparison problem of standard Gini means (cf. [5,8,32]):

Theorem 12. *Let $p, q, r, s \in \mathbb{R}$. Then the comparison inequality*

$$G_{p,q}(x,y) \leqslant G_{r,s}(x,y) \quad (x, y \in \mathbb{R}_+)$$

holds if and only if the conditions

$$p + q \leqslant r + s, \qquad \alpha(p,q) \leqslant \alpha(r,s) \quad \text{and} \quad \beta(p,q) \leqslant \beta(r,s) \tag{34}$$

are satisfied.

For the comparison of Stolarsky means, we have (cf. [6,8,17,31]):

Theorem 13. *Let $p, q, r, s \in \mathbb{R}$. Then the comparison inequality*

$$S_{p,q}(x, y) \leqslant S_{r,s}(x, y) \quad (x, y \in \mathbb{R}_+)$$

holds if and only if the conditions

$$p + q \leqslant r + s, \qquad \alpha(p, q) \leqslant \alpha(r, s) \quad \text{and} \quad \gamma(p, q) \leqslant \gamma(r, s) \tag{35}$$

are satisfied.

Theorems 12 and 13 completely solve the comparison problem of generalized Gini means for the particular measures $\mu = \frac{1}{2}(\delta_0 + \delta_1)$ and $\mu = \lambda$. The mixed comparison problem of Gini and Stolarsky means was studied in the papers [27] and [9] where various necessary as well as sufficient conditions were obtained. For the comparison of generalized Gini means with two fixed measures, we obtain, as a direct consequence of Theorem 5, the following necessary condition.

Theorem 14. *Let $p, q, r, s \in \mathbb{R}$ and μ, ν be Borel probability measures on $[0, 1]$ and let $I \subset \mathbb{R}_+$ be an open subinterval. Suppose that*

$$G_{p,q;\mu}(x, y) \leqslant G_{r,s;\nu}(x, y) \quad (x, y \in I) \tag{36}$$

holds. Then

$$\mu_1 = \nu_1 \tag{37}$$

and

$$\left(\mu_2 - \mu_1^2\right)(p + q - 1) \leqslant \left(\nu_2 - \nu_1^2\right)(r + s - 1). \tag{38}$$

Conversely, if (37) *is valid and* (38) *is satisfied with strict inequality sign, then* (36) *holds for $x, y \in I$ provided that $|x - y|$ is small enough.*

Proof. If f, g are defined by (33), then the function $\mathcal{D}_{f,g}$ is of the form

$$\mathcal{D}_{f,g}(x, y) = \Delta_{p,q}(x, y) := y^{p+q}\delta_{p,q}\left(\frac{x}{y}\right) \quad (x, y \in \mathbb{R}_+), \tag{39}$$

where

$$\delta_{p,q}(t) := \begin{cases} \frac{t^p - t^q}{p-q} & \text{if } p \neq q, \\ t^p \ln t & \text{if } p = q \end{cases} \quad (t \in \mathbb{R}_+). \tag{40}$$

Therefore,

$$\frac{\partial_1^2 \mathcal{D}_{f,g}(x, x)}{\partial_1 \mathcal{D}_{f,g}(x, x)} = \frac{\delta''_{p,q}(1)}{\delta'_{p,q}(1)}\frac{1}{x} = (p + q - 1)\frac{1}{x} \quad (x \in \mathbb{R}_+).$$

This formula and the direct application of Theorem 5 yields that (37) and (38) are necessary conditions for the comparison inequality (36) to hold. □

Corollary 15. *Let $p, q, r, s \in \mathbb{R}$ and μ be a Borel probability measure on $[0, 1]$ such that $\mu_2 - \mu_1^2 \neq 0$ and let $I \subset \mathbb{R}_+$ be an open subinterval. If*

$$G_{p,q;\mu}(x, y) \leqslant G_{r,s;\mu}(x, y) \quad (x, y \in I) \tag{41}$$

holds, then

$$p + q \leqslant r + s. \tag{42}$$

Conversely, if (42) *is satisfied with strict inequality sign, then* (41) *holds for $x, y \in I$ provided that $|x - y|$ is small enough.*

Theorem 16. *Let $p, q, r, s \in \mathbb{R}$. The following three assertions are equivalent:*

(i) *for all Borel probability measures μ on $[0, 1]$,*

$$G_{p,q;\mu}(x, y) \leqslant G_{r,s;\mu}(x, y) \quad (x, y \in \mathbb{R}_+); \tag{43}$$

(ii) *for all* $i \in \mathbb{N}$

$$G_{p,q;m_{\frac{1}{i}}}(x,y) \leqslant G_{r,sm_{\frac{1}{i}}}(x,y) \quad (x,y \in \mathbb{R}_+) \tag{44}$$

(*where* (m_s) *is the one-parameter family of measures defined by* (8));

(iii)

$$\min(p,q) \leqslant \min(r,s) \quad \textit{and} \quad \max(p,q) \leqslant \max(r,s). \tag{45}$$

Proof. Using the notation introduced in (39), (40), by Theorem 6, conditions (i) and (ii) are equivalent to the inequality

$$\frac{\Delta_{p,q}(v,u)}{\partial_1 \Delta_{p,q}(u,u)} \leqslant \frac{\Delta_{r,s}(v,u)}{\partial_1 \Delta_{r,s}(u,u)} \quad (u,v \in \mathbb{R}_+),$$

which, with the notation $t = v/u$, can be simplified to

$$\delta_{p,q}(t) \leqslant \delta_{r,s}(t) \quad (t \in \mathbb{R}_+). \tag{46}$$

This inequality is known to be equivalent to condition (45) (cf. [12,33]). □

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